5 Bifurcation I

♢ **5.1.** Consider the following equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \alpha, \quad r, K, \alpha > 0.$$

(This equation is the simplest model of a population under harvesting α). Considering α as a bifurcation parameter, find the point of the fold bifurcation and check the conditions of nondegeneracity.

♢ **5.2.** The condition for the equilibrium to be nonhyperbolic is clearly necessary for an equlibrium bifurcation. Is it also sufficient?

♢ **5.3.** Draw a bifurcation diagram for

$$\dot{x} = \lambda^2 + 2a\lambda x + x^2$$

for a fixed and λ being the bifurcation parameter. Notice the difference between $|a| < 1$ and $|a| > 1$.

♢ **5.4.** Draw a bifurcation diagram for

$$\dot{x} = 1 + \mu + 2\mu x - x^3.$$

♢ **5.5.** Recall that we proved in class that if a one-dimensional dynamical system $\dot{x} = f(x, \alpha)$ satisfies the bifurcation conditions $f(0, 0) = 0$ and $f'_x(0, 0) = 0$ and nondegeneracity conditions $f''_{xx}(0, 0) \neq 0$, $f'_\alpha(0, 0) \neq 0$ then we have the generic fold bifurcation. I explained why it is the only generic codim 1 bifurcation in one-dimensional systems. These explanations notwithstanding, from a purely practical point of view some of these nondegeracity conditions can be violated and hence we have different (non *generic*) bifurcation. In particular, you are asked in this exercise to analyze the following two cases:

1. Assume that
$$f'_\alpha(0, 0) = 0, \quad f''_{x,\alpha}(0, 0) \neq 0, \quad f''_{xx}(0, 0) \neq 0$$
and find the corresponding *normal form* of this bifurcation. (Hint: assume that $f(x, \alpha) = xF(x, \alpha)$.) You may want to start with analysis of $\dot{x} = \alpha x \pm x^2$. This is called *transcritical bifurcation.*

2. Assume that
$$f'_\alpha(0, 0) = 0, \quad f''_{xx}(0, 0) = 0, \quad f''_{\alpha x}(0, 0) \neq 0, \quad f'''_{xxx}(0, 0) \neq 0$$
and find the corresponding *normal form* of this bifurcation. (Hint: see the hint above.) You may want to start with analysis of $\dot{x} = \alpha x \pm x^3$. This is called *pitchfork* bifurcation.

♢ **5.6.** Provide a full analysis of the Hopf bifurcation in Rayleigh's equation:

$$\ddot{x} - \dot{x}^3 - 2\alpha\dot{x} + x = 0.$$

♢ **5.7.** Provide a full analysis of the Hopf bifurcation in Van der Pol's equation:

$$\ddot{x} - (\alpha - x^2)\dot{x} + y = 0.$$

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$\diamondsuit$ **5.8.** Provide a full analysis of the Hopf bifurcation in the Brusselator:

$$\begin{aligned}\dot{x}_1 &= A - (B+1)x_1 + x_1^2 x_2,\\ \dot{x}_2 &= Bx_1 - x_1^2 x_2,\end{aligned}$$

for a fixed $A > 0$ and B is the bifurcation parameter.

Hint: you should use some software for symbolic calculations. The first Lyapunov value is

$$-\frac{2+A^2}{2A(1+A^2)}\,.$$

$\diamondsuit$ **5.9.** Study the dynamics near the origin of

$$\dot{x} = -y - y^3, \quad \dot{y} = 2x.$$

The same question is about

$$\dot{x} = y + x^2, \quad \dot{y} = -y - x^2.$$

$\diamondsuit$ **5.10.** Show that the origin of

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - xz,\\ \dot{z} &= -z + \alpha x^2\end{aligned}$$

is asymptotically stable if $\alpha < 0$ and unstable if $\alpha > 0$.

$\diamondsuit$ **5.11.** To check the transversally condition for n-dim Hopf bifurcation one can use the formula

$$\mu'(0) = \operatorname{Re}\langle p, A'(0)q\rangle,$$

where all the notation as in class. Prove it.